TL;DR

The Herschel–Bulkley model unifies Newtonian, Bingham, and power-law fluids via a yield stress and a strain-rate-dependent viscosity. An ϵ-regularization ensures stable computations and recovers simpler models (Newtonian, Bingham) by tuning model parameters. Dimensionless groups (e.g., the plasto-capillary number and the effective Ohnesorge) capture the interplay of fluid rheology, capillarity, and flow scales. Implementation details are provided, along with references, open-source code, and demonstrations of bubble-burst simulations in viscoplastic media.

Features:

- Yield stress τ_y
- Power law dependance on the strain rate
  - Shear thinning for $n < 1$.
  - Shear thickening for $n > 1$.
- Bingham model for $n = 1$.
- Newtonian fluid for $n = 1$ and $\tau_y = 0$.

ε-formulation

$$\boldsymbol{\tau} = \tau_y \boldsymbol{\mathcal{I}} \ + \ K(2\boldsymbol{\mathcal{D}})^n = 2\left[\frac{\tau_y}{2\|\boldsymbol{\mathcal{D}}\| + \varepsilon}\boldsymbol{\mathcal{I}} + K\left(2\|\boldsymbol{\mathcal{D}}\| + \epsilon\right)^{n-1}\right]\boldsymbol{\mathcal{D}}.$$

Normalizing stresses with γ/R_0, length with R_0, and velocity with $\sqrt{\gamma/\rho_l R_0}$...

$$\tilde{\boldsymbol{\tau}} = 2\left[\frac{\mathcal{J}}{2\|\tilde{\boldsymbol{\mathcal{D}}}\| + \varepsilon}\boldsymbol{\mathcal{I}} + Oh_K\left(2\|\tilde{\boldsymbol{\mathcal{D}}}\| + \epsilon\right)^{n-1}\right]\tilde{\boldsymbol{\mathcal{D}}}.$$

Here, the effective Ohnesorge is

$$Oh_K = \frac{K}{\sqrt{\rho_l^n \gamma^{2-n} R_0^{3n-2}}}$$

The plasto-capillary number $\mathcal{J}$ is $\mathcal{J} = \frac{\tau_y R_0}{\gamma}$

One can easily see that putting $n = 1$ recovers the Bingham model with $Oh = \eta_l/\sqrt{\rho_l \gamma R_0}$. Additionally, with $n = 1$ & $\mathcal{J}$ = 0, the model will give a `Newtonian` response.

More details on the implementation

Calculate the norm of the deformation tensor $\mathcal{D}$:

$\mathcal{D}_{11} = \frac{\partial u_r}{\partial r}\ \mathcal{D}_{22} = \frac{u_r}{r}\ \mathcal{D}_{13} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\$\$\$\$\mathcal{D}_{31} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)\ \mathcal{D}_{33} = \frac{\partial u_z}{\partial z}\ \mathcal{D}_{12} = \mathcal{D}_{23} = 0.$
The second invariant is $\mathcal{D}_2 = \sqrt{\mathcal{D}_{ij}\mathcal{D}_{ij}}$ (this is the Frobenius norm)

$\mathcal{D}_2^2 = \mathcal{D}_{ij}\mathcal{D}_{ij} = \mathcal{D}_{11}\mathcal{D}_{11} + \mathcal{D}_{22}\mathcal{D}_{22} + \mathcal{D}_{13}\mathcal{D}_{31} + \mathcal{D}_{31}\mathcal{D}_{13} + \mathcal{D}_{33}\mathcal{D}_{33}$

Note: $\|\mathcal{D}\| = D_2/\sqrt{2}$.

We use the formulation as given in **Balmforth et al. (2013)** [1], who use the strain rate tensor $\dot{\mathcal{S}}$ which and its norm $\sqrt{\frac{1}{2}\dot{\mathcal{S}}_{ij}\dot{\mathcal{S}}_{ij}}$. Of course, given $\dot{\mathcal{S}}_{ij} = 2D_{ij}$.

Calculate the equivalent viscosity

Factorizing with $2\mathcal{D}_{ij}$ to obtain an equivalent viscosity

$$+Oh_K\left(2\|\tilde{\mathcal{D}}\| + \epsilon\right)^{n-1}$$

In this formulation, ε is a small number to ensure numerical stability. The term $\frac{\tau_y}{\varepsilon} + \ldots$ is equivalent to the μ_{max} of the previous (v1.0, see: **GitHub**) formulation [2].

Note: The fluid flows always, it is not a solid, but a very viscous fluid.

Reproduced from: **P.-Y. Lagrée's Sandbox**. Here, we use a face implementation of the regularisation method, described **here**.

Further exploration:

Video showcasing a typical simulation of bubble bursting in a Herschel–Bulkley fluid medium

[YouTube video player]

[Open on YouTube]

More resources

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References

[1] N. J. Balmforth, I. A. Frigaard, and G. Ovarlez, "Yielding to Stress: Recent Developments in Viscoplastic Fluid Mechanics," *Annu. Rev. Fluid Mech.*, vol. 46, pp. 121–146, Jan. 2014, doi: **10.1146/annurev-fluid-010313-141424**.

[2] V. Sanjay, D. Lohse, and M. Jalaal, "Bursting bubble in a viscoplastic medium," *J. Fluid Mech.*, vol. 922, p. A2, 2021.

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